GEODESICALLY REVERSIBLE FINSLER 2-SPHERES OF CONSTANT CURVATURE

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ABSTRACT. A Finsler space (M, Σ) is said to be *geodesically reversible* if each oriented geodesic can be reparametrized as a geodesic with the reverse orientation. A reversible Finsler space is geodesically reversible, but the converse need not be true.

In this note, building on recent work of LeBrun and Mason [15], it is shown that a geodesically reversible Finsler metric of constant flag curvature on the 2-sphere is necessarily projectively flat.

As a corollary, using a previous result of the author [5], it is shown that a reversible Finsler metric of constant flag curvature on the 2-sphere is necessarily a Riemannian metric of constant Gauss curvature, thus settling a long-standing problem in Finsler geometry.

1. Introduction

The purpose of this note is to settle a long-standing problem in Finsler geometry: Whether there exists a *reversible* Finsler metric on the 2-sphere with constant flag curvature that is not Riemannian. By making use of some old results and a fundamental new result of LeBrun and Mason, I show that such Finsler structures do not exist.

First, I prove something related: Any geodesically reversible Finsler metric on the 2-sphere with constant flag curvature must be projectively flat. Since the projectively flat Finsler metrics with constant flag curvature on S^2 were classified some years ago [5], the above result then reduces to examining the Finsler structures provided by this classification.

In a famous 1988 paper [1], Akbar-Zadeh showed that a (not necessarily reversible) Finsler structure on a compact surface with constant negative flag curvature was necessarily Riemannian or with zero flag curvature was necessarily a translation-invariant Finsler structure on the standard 2-torus $\mathbb{R}^2/\mathbb{Z}^2$. This naturally raised the question about what happens in the case of constant positive flag curvature.

This problem was made more interesting by the discovery of non-reversible Finsler metrics on the 2-sphere with constant positive flag curvature in [4]. (However, it should be pointed out that Katok had already constructed non-reversible

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Finsler metrics on the 2-sphere [20] that later turned out to have constant flag curvature, although, apparently, this was not known at the time of [4].)

In the interests of brevity, no attempt has been made to give an exposition of the basics of Finsler geometry. There are many sources for this background material however, among them [2], [8], [9, 10], and [16].

For background more specifically suited for studying the case of constant flag curvature, including its proper formulation in higher dimensions, see [3], [12, 13, 14], and [17, 18, 19].

The corresponding question about (geodesically) reversible Finsler metrics of constant positive flag curvature on the n-sphere for $n > 2$ remains open at this writing, since an essential component of the proof for $n = 2$ that is due to LeBrun and Mason has not yet been generalized to higher dimensions.

Contents

2. Structure equations

In this section, Cartan's structure equations for a Finsler surface will be recalled.

2.1. **Cartan's coframing.** Let M be a surface and let $\Sigma \subset TM$ be a smooth Finsler structure. I.e., Σ is a smooth hypersurface in M such that the basepoint projection $\pi : \Sigma \to M$ is a surjective submersion and such that each fiber

$$\pi^{-1}(x) = \Sigma_x = \Sigma \cap T_xM \tag{2.1}$$

is a smooth, strictly convex curve in T_xM whose convex hull contains the origin 0_x in its interior.

Remark 1 (Reversibility). Note that there is no assumption that $\Sigma = -\Sigma$. In other words, a Finsler structure need not be 'reversible' (some sources call this property 'symmetry'), and assumption is not needed for the development of the local theory.

One should think of Σ as the unit vectors of a 'Finsler metric', i.e., a function $F : TM \to \mathbb{R}$ that restricts to each tangent space T_xM to be a not-necessarily-symmetric but strictly convex Banach norm on T_xM.

2.1.1. *Σ-length of oriented curves.* A curve $\gamma : (a,b) \to M$ will be said to be a Σ-curve (or 'unit speed curve') if $\gamma'(t)$ lies in Σ for all $t \in (a,b)$. Any smooth, immersed curve $\gamma : (a,b) \to M$ has an orientation-preserving reparametrization $h : (u,v) \to (a,b)$ such that $\gamma \circ h$ is a Σ-curve. This reparametrization is unique up to translation in the domain of h. Thus, one can unambiguously define the (oriented) Σ-length of a subcurve $\gamma : (\alpha,\beta) \to M$ to be $h^{-1}(\beta) - h^{-1}(\alpha)$, when $a < \alpha < \beta < b$.

2.1.2. *Cartan's coframing.* The fundamental result about the geometry of Finsler surfaces is due to Cartan [7]:

Theorem 1 (Canonical coframing)**.** *Let $\Sigma \subset TM$ be a Finsler structure on the oriented surface M with basepoint projection $\pi : \Sigma \to M$. Then there exists a unique coframing $\big(\omega_1, \omega_2, \omega_3\big)$ on Σ with the properties:*

1. *$\omega_1{\wedge}\omega_2$ is a positive multiple of any π-pullback of a positive 2-form on M,*
2. *The tangential lift γ' of any Σ-curve satisfies $(\gamma')^*\omega_2 = 0$ and $(\gamma')^*\omega_1 = dt$,*
3. *$\mathrm{d}\omega_1{\wedge}\omega_2 = 0$,*
4. *$\omega_1{\wedge}\mathrm{d}\omega_1 = \omega_2{\wedge}\mathrm{d}\omega_2$, and*
5. *$\mathrm{d}\omega_1 = \omega_3{\wedge}\omega_2$ and $\omega_3{\wedge}\mathrm{d}\omega_2 = 0$.*

Moreover, there exist unique functions I, J, and K on Σ so that

$$
\begin{aligned}
\mathrm{d}\omega_1 &= -\omega_2 \wedge \omega_3, \\
\mathrm{d}\omega_2 &= -\omega_3 \wedge \big(\omega_1 - I\,\omega_2\big), \\
\mathrm{d}\omega_3 &= -\big(K\,\omega_1 - J\,\omega_3\big) \wedge \omega_2.
\end{aligned}
\tag{2.2}
$$

Remark 2 (The invariants I, J, and K.). The 1-form ω_1 is called *Hilbert's invariant integral*. A Σ-curve γ is a geodesic of the Finsler structure if and only if its tangential lift satisfies $(\gamma')^*\omega_3 = 0$. (Of course, by definition, $(\gamma')^*\omega_2 = 0$.)

The function I vanishes if and only if Σ is the unit circle bundle of a Riemannian metric on M, in which case the function K becomes the π-pullback of the Gauss curvature of the underlying metric.

The function J vanishes if and only if the Finsler structure is what is called *Landsberg* [2].

The function K is known as the Finsler-Gauss curvature and plays the same role in the Jacobi equation for Finsler geodesics as the Gauss curvature does in the Jacobi equation for Riemannian geodesics.

Let X_1, X_2, and X_3 be the vector fields on Σ that are dual to the coframing $(\omega_1, \omega_2, \omega_3)$. Then the flow of X_1 is the geodesic flow on Σ.

Remark 3 (The effect of orientations). If one reverses the orientation of M, then the canonical coframing ω on Σ is replaced by $(\omega_1, -\omega_2, -\omega_3)$.

In fact, Cartan's actual statement of Theorem 1 does not assume that M is oriented and concludes that there is a canonical coframing on Σ up to the sign ambiguity given here. The present version of the statement is a trivial rearrangement of Cartan's that is more easily applied in the situations encountered in this note.

2.1.3. *Reconstruction of M and its Finsler structure.* The information contained in the 3-manifold Σ and its coframing $\omega = (\omega_1, \omega_2, \omega_3)$ is sufficient to recover M, its orientation, and the embedding of Σ into M, a fact that is implicit in Cartan's analysis:

Proposition 1 (Isometries and automorphisms)**.** *For any orientation-preserving Finsler isometry $\phi : M \to M$, its derivative $\phi' : TM \to TM$ induces a diffeomorphism $\phi' : \Sigma \to \Sigma$ that preserves the coframing $\omega = (\omega_1, \omega_2, \omega_3)$.*

Conversely, any diffeomorphism $\psi : \Sigma \to \Sigma$ that preserves ω is of the form $\psi = \phi'$ for a unique orientation-preserving Finsler isometry $\phi : M \to M$.

Proof. The first statement follows directly from Theorem 1. I will sketch how the converse goes.

The integral curves of the system $\omega_1 = \omega_2 = 0$ on Σ are closed and the codimension 2 foliation they define has trivial holonomy, so M can be identified with the leaf space of this system and carries a unique smooth structure for which the leaf projection $\pi : \Sigma \to M$ is a smooth submersion.

Because of the connectedness of the π-fibers, there will be a unique orientation on M such that a positive 2-form pulls back under π to be a positive multiple of $\omega_1 \wedge \omega_2$. Thus, M, its smooth structure, and its orientation can be recovered from the coframing.

The inclusion $\iota : \Sigma \to TM$ is then seen to be simply given by $\iota(u) = \pi'\bigl(X_1(u)\bigr) \in T_{\pi(u)}M$. Thus, even the Finsler structure on M can be recovered from Σ and the coframing.

The desired result now follows by noting that any $\psi : \Sigma \to \Sigma$ that preserves ω will necessarily preserve the integral curves of the system $\omega_1 = \omega_2 = 0$ and hence induce a map $\phi : M \to M$ that is π-intertwined with ψ. The verification that ϕ is an orientation-preserving Finsler isometry is easy and can be left to the reader. $\square$

Corollary 1 (Orientation-reversing isometries)**.** *Any diffeomorphism $\psi : \Sigma \to \Sigma$ that satisfies $\psi^*(\omega) = (\omega_1, -\omega_2, -\omega_3)$ is of the form $\psi = \phi'$ for a unique orientation-*reversing *Finsler isometry $\phi : M \to M$.*

2.2. **Bianchi identities.** Taking the exterior derivatives of the structure equations (2.2) yields the formulae

$$\begin{pmatrix} \mathrm{d}I \\ \mathrm{d}J \\ \mathrm{d}K \end{pmatrix} = \begin{pmatrix} J & I_2 & I_3 \\ -K_3 - KI & J_2 & J_3 \\ K_1 & K_2 & K_3 \end{pmatrix} \begin{pmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \end{pmatrix} \tag{2.3}$$

for some functions I_2, I_3, J_2, J_3, K_1, K_2, and K_3 on Σ.

2.3. **Simplifications when $K \equiv 1$.** The Finsler structures of interest in this article are the ones that satisfy $K \equiv 1$. In this case, the structure equations simplify to

$$\begin{aligned} \mathrm{d}\omega_1 &= -\omega_2 \wedge \omega_3, \\ \mathrm{d}\omega_2 &= -\omega_3 \wedge \bigl(\omega_1 - I\,\omega_2\bigr), \\ \mathrm{d}\omega_3 &= -\bigl(\omega_1 - J\,\omega_3\bigr) \wedge \omega_2, \end{aligned} \tag{2.4}$$

and the Bianchi identities become

$$\begin{pmatrix} \mathrm{d}I \\ \mathrm{d}J \end{pmatrix} = \begin{pmatrix} J & I_2 & I_3 \\ -I & J_2 & J_3 \end{pmatrix} \begin{pmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \end{pmatrix}. \tag{2.5}$$

Remark 4 (A geodesic conservation law). The equations (2.5) imply that the function $I^2 + J^2$ is constant on the integral curves of $\omega_2 = \omega_3 = 0$, i.e., the lifts of geodesics. This function need not be constant on Σ, in which case, it provides a

nontrivial conservation law for the geodesic flow on Σ. (Of course, this function vanishes identically in the Riemannian case.)

2.4. **Some global consequences of $K \equiv 1$.** Suppose now that M is connected and geodesically complete, i.e., that, the vector field X_1 is complete on Σ (in both forward and backward time). Of course, if M were assumed to be compact, then Σ would be also, and the completeness of X_1 would follow from this.

The assumption that M be connected implies that Σ is connected.

Let $\Psi : \Sigma \times \mathbb{R} \to \Sigma$ be the flow of X_1 and, for brevity, let $\Psi_t : \Sigma \to \Sigma$ denote the time t flow of X_1. Since the structure equations imply

$$\mathsf{L}_{X_1}\,\omega_1 = 0, \qquad \mathsf{L}_{X_1}\,\omega_2 = \omega_3, \qquad \mathsf{L}_{X_1}\,\omega_3 = -\omega_2, \tag{2.6}$$

it follows (letting $t : \Sigma \times \mathbb{R} \to \mathbb{R}$ denote the coordinate that is the projection on the second factor) that

$$\begin{aligned} \Psi^*\omega_1 &= \omega_1 + \mathrm{d}t, \\ \Psi^*\omega_2 &= \quad \cos t\,\omega_2 + \sin t\,\omega_3, \\ \Psi^*\omega_3 &= -\sin t\,\omega_2 + \cos t\,\omega_3. \end{aligned} \tag{2.7}$$

Proposition 2 (The quasi-antipodal map)**.** *There exists a unique orientation-reversing Finsler isometry $\alpha : M \to M$ such that $\alpha' = \Psi_\pi$. For any point $p \in M$, every unit speed geodesic leaving p passes through $\alpha(p)$ at distance π.*

Proof. By (2.7), it follows that $\Psi_\pi : \Sigma \to \Sigma$ satisfies

$$\Psi_\pi^*\omega = (\omega_1, -\omega_2, -\omega_3). \tag{2.8}$$

Hence, by Corollary 1, there is a unique orientation-reversing Finsler isometry $\alpha : M \to M$ such that $\Psi_\pi = \alpha' : \Sigma \to \Sigma$.

Since X_1 is the geodesic flow vector field, any unit speed geodesic leaving p at time 0 is of the form $\gamma(t) = \pi\bigl(\Psi_t(u)\bigr)$ for some $u \in \Sigma_p \subset T_pM$. Thus, $\gamma(\pi) = \pi\bigl(\Psi_\pi(u)\bigr) = \pi\bigl(\alpha'(u)\bigr) = \alpha(p)$, as claimed. $\square$

Now, for any fixed $p \in M$, the fiber $\Sigma_p \subset T_pM$, is diffeomorphic to a circle and is naturally oriented by taking the pullback of ω_3 to Σ_p to be a positive 1-form. Define $r(p) > 0$ by

$$r(p) = \frac{1}{2\pi}\int_{\Sigma_p} \omega_3\,. \tag{2.9}$$

Then Σ_p can be parametrized by a mapping $\iota_p : S^1 \to \Sigma_p$ that satisfies $\iota_p^*(\omega_3) = r(p)\,\mathrm{d}\theta$ and that is uniquely determined once one fixes $\iota_p(0) = u \in \Sigma_p$. Such a parametrization ι_p will be referred to as an *angle measure* on Σ_p.

Proposition 3 (Geodesic polar coordinates)**.** *For any $p \in M$, fix an angle measure $\iota_p : S^1 \to \Sigma_p$. Then the mapping $E_p : S^2 \to M$ defined by*

$$E_p(\sin t\cos\theta, \sin t\sin\theta, \cos t) = \pi\bigl(\Psi_t(\iota_p(\theta))\bigr) \tag{2.10}$$

is an orientation-preserving homeomorphism that is smooth away from $(0, 0, \pm 1) \in S^2$. In particular, M is homeomorphic to the 2-sphere and its diameter as a Finsler space is equal to π.

Proof. Consider the mapping $R_p : S^1 \times \mathbb{R} \to \Sigma$ defined by

$$R_p(\theta,t) = \Psi\bigl(\iota_p(\theta),t\bigr). \tag{2.11}$$

The formulae (2.7), the fact that Ψ is the flow of X_1, and the defining property of ι_p then combine to show that

$$R_p^*(\omega_1\wedge\omega_2) = \mathrm{d}t\wedge\bigl(\sin t\ r(p)\,\mathrm{d}\theta\bigr) = r(p)\,\sin t\,\mathrm{d}t\wedge\mathrm{d}\theta. \tag{2.12}$$

Thus, the composition $\pi\circ R_p : S^1\times\mathbb{R} \to M$ is a smooth map that is a local diffeomorphism away from the circles $(\theta,t) = (\theta,k\pi)$ for each integer k. Of course, $\pi\bigl(R_p(\theta,0)\bigr) = p$ and $\pi\bigl(R_p(\theta,\pi)\bigr) = \alpha(p)$ for all $\theta\in S^1$.

It now follows that the formula (2.10) well-defines a mapping $E_p : S^2 \to M$ that is smooth and an orientation-preserving local diffeomorphism away from $(0,0,\pm1)$. Near the two points $(0,0,\pm1)$, the mapping E_p is still a (not necessarily differentiable) orientation-preserving local homeomorphism.

It follows that $E_p : S^2\to M$ is a topological covering map. Since M is orientable by assumption, it follows that E_p must be a homeomorphism and, in particular, must be one-to-one and onto. The statement about diameters follows. □

Remark 5. Versions of Propositions 2 and 3 were proved by Shen [17] in the case that Σ is reversible (see Definition 1).

Proposition 4. *Either* $\alpha^2 = \mathrm{id}$ *on* M (*in which case, all of the* Σ*-geodesics are closed of length* 2π) *or else* α^2 *has exactly two fixed points, say* n *and* $\alpha(n)$.

In the latter case, there exists a positive definite inner product on T_nM *that is invariant under* $(\alpha^2)'(n) : T_nM \to T_nM$ *and there is an angle* $\theta_n\in(0,2\pi)$ *such that* $(\alpha^2)'(n)$ *is a counterclockwise rotation by* θ_n *in this inner product.*

Proof. Assume that $\alpha^2 : M\to M$ is not the identity, or else there is nothing to prove. Since α^2 is an orientation preserving diffeomorphism of the 2-sphere, it must have at least one fixed point. Let n be such a fixed point. By the very definition of α, it then follows that $\alpha(n)$ is also a fixed point of α^2. It must be shown that α^2 has no other fixed points.

First, consider the linear map $L = (\alpha^2)'(n) : T_nM \to T_nM$. Since α^2 is a Finsler isometry, the linear map L must preserve $\Sigma_n \subset T_nM$. Let $K_n\subset T_nM$ be the convex set bounded by Σ_n.

Define a positive definite quadratic form on T^*_nM by letting $\langle\lambda_1,\lambda_2\rangle$ be defined for $\lambda_1,\lambda_2\in T^*_nM$ to be the average of the quadratic function $\lambda_1\lambda_2$ over K_n (using any translation invariant measure on K_n induced by its inclusion into the vector space T_nM). Since L is a linear map carrying K_n into itself, it must preserve this quadratic form and hence must also preserve the dual (positive-definite) quadratic form on T_nM. Since L also preserves an orientation on T_nM, it follows that, with respect to this invariant inner product, L must be a counterclockwise rotation by some angle $\theta_n\in[0,2\pi)$.

If θ_n were 0, i.e., L were the identity on T_nM, then all of the geodesics through n would close at length 2π. In particular, the mapping $\Psi_{2\pi} : \Sigma\to\Sigma$ would have a fixed point and would preserve the coframing ω, implying that $\Psi_{2\pi}$ is the identity on Σ and hence that α^2 would be the identity. Thus, $0<\theta_n<2\pi$.

Since n was an arbitrarily chosen fixed point of α^2, it follows that every fixed point of α^2 is an isolated elliptic fixed point, i.e., a fixed point of index 1. Since M is diffeomorphic to S^2, the Hopf Index Theorem implies that the map α^2 has exactly two fixed points. Thus α^2 has no fixed points other than n and $\alpha(n)$. □

Remark 6 (The Katok examples). The Katok examples analyzed by Ziller [20] turn out[1] to have $K \equiv 1$ and are examples in which α^2 is not the identity. Thus, the second possibility in Proposition 4 does occur.

In any case, when α^2 is not the identity, $\theta_n + \theta_{\alpha(n)} = 2\pi$.

If the angle θ_n defined in Proposition 4 is not a rational multiple of π, then the iterates of α^2 are dense in a circle of Finsler isometries of (M, Σ) that fix n and $\alpha(n)$. In such a case, (M, Σ) is rotationally symmetric about n. Moreover, it is symmetric (in an orientation reversing sense) with respect to α.

If $\theta_n = 2\pi(p/q)$ where $0 < p \le q$ and p and q have no common factors, then α^{2q} is the identity, so that every geodesic closes at length $2\pi q$ (though some may close sooner).

3. A double fibration

Throughout this section Σ will be assumed to be a Finsler structure on M (assumed diffeomorphic to the 2-sphere) satisfying $K \equiv 1$.

I begin by noting that, if all the geodesics on M close at distance 2π, then the set of oriented Σ-geodesics has the structure of a manifold in a natural way.

Proposition 5 (The space of oriented geodesics). *If α^2 is the identity, then the action*

$$u \cdot \mathrm{e}^{\mathrm{i}t} = \Psi(u, t) \tag{3.1}$$

defines a smooth, free S^1-action on Σ whose orbits are the integral curves of $\omega_2 = \omega_3 = 0$ and there exists a smooth surface Λ diffeomorphic to S^2 and a smooth submersion $\lambda : \Sigma \to \Lambda$ so that the action (3.1) *makes $\lambda : \Sigma \to \Lambda$ into a principal right S^1-bundle over Λ.*

Proof. If α^2 is the identity, then the flow of X_1 is periodic of period 2π, so (3.1) defines a smooth S^1-action on Σ. Since X_1 never vanishes, this action has no fixed points. Thus, if this action were not free, then there would be a $u \in \Sigma$ and an integer $k \ge 2$ such that $\Psi(u, 2\pi/k) = u$. However, since $0 < 2\pi/k \le \pi$, the equality $\Psi(u, 2\pi/k) = u$ would violate Proposition 3, since then $E_{\pi(u)} : S^2 \to M$ could not be one-to-one.

Thus, the S^1-action (3.1) is free and the rest of the proposition follows by standard arguments. □

Remark 7 (Double fibration and path geometries). The two mappings $\pi : \Sigma \to M$ and $\lambda : \Sigma \to \Lambda$ define a double fibration and it is easy to see that this double fibration satisfies the usual nondegeneracy axioms for double fibrations. For example, $\lambda \times \pi : \Sigma \to \Lambda \times M$ is clearly a smooth embedding. The other properties are similarly easy to verify using the structure equations. Thus, Σ defines a (generalized) path geometry on each of Λ and M.

For more background on path geometries and their invariants, see, for example, Section 2 of [5].

[1] Colleen Robles, private communication

3.1. **Induced structures on Λ.** I will now recall some results from [5]. Throughout this subsection, I will be assuming that α^2 is the identity, so that Λ exists as a smooth manifold.

The relations (2.7) show that the quadratic form ${\omega_2}^2 + {\omega_3}^2$ is invariant under the flow of X_1. Consequently, there is a unique Riemannian metric on Λ, say g, such that

$$\lambda^*(g) = {\omega_2}^2 + {\omega_3}^2 \tag{3.2}$$

Moreover, the 2-form $\omega_3{\wedge}\omega_2$ is invariant under the flow of X_1, so it is the pullback under λ of an area 2-form for g, which will be denoted $\mathrm{d}A_g$.

Now, there is an embedding $\xi : \Sigma \to T\Lambda$ defined by

$$\xi(u) = \lambda'\bigl(X_3(u)\bigr) \tag{3.3}$$

and one sees that ξ embeds Λ as the unit sphere bundle of Λ endowed with the metric g.

The structure equations (2.4) show that, under this identification of Σ with the unit sphere bundle of Λ, the Levi-Civita connection form on Σ is

$$\rho = -\omega_1 + I\,\omega_2 + J\,\omega_3\,. \tag{3.4}$$

Note that $-\omega_1$ and $I\,\omega_2 + J\,\omega_3$ are invariant under the flow of X_1.

For the next two results, which follow from the structure equations derived so far by simply unraveling the definitions, the reader may want to consult LeBrun and Mason [15] for the definition and properties of the projective structure associated to an affine connection on a surface. [They restrict themselves to the consideration of torsion-free connections, but, as they point out, this does not affect the results.]

Proposition 6. *There exists a g-compatible affine connection ∇ on Λ such that the ∇-geodesics are the λ-projections of the integral curves of $\omega_1 = \omega_2 = 0$.* □

Corollary 2. *The geodesics of the projective structure $[\nabla]$ on Λ are closed.*

Proof. By Proposition 6, the geodesics of $[\nabla]$ are the λ-projections of the integral curves of the system $\omega_1 = \omega_2 = 0$, but these integral curves are the fibers of the map $\pi : \Sigma \to M$ and hence are closed. □

3.2. **Geodesic reversibility implies geodesic periodicity.** It is now time to come to the main point of this note.

Definition 1 (Reversibility). The Finsler structure $\Sigma \subset TM$ is said to be *reversible* if $\Sigma = -\Sigma$.

Definition 2 (Geodesic reversibility). A Finsler structure $\Sigma \subset TM$ will be said to be *geodesically reversible* if any Σ-geodesic $\gamma : (a,b) \to TM$ can be reparametrized in an orientation-reversing way so as to remain a Σ-geodesic.

Remark 8. Any reversible Finsler structure is geodesically reversible. On the other hand, the non-Riemannian Finsler examples constructed in Section 4 of [5] are geodesically reversible but not reversible, so the reverse implication does not hold.

Proposition 7. *If (M,Σ) is geodesically reversible, then α^2 is the identity on M.*

Proof. For any point $p \in M$, consider the geodesics leaving p. By Proposition 3, they all converge at distance π on $\alpha(p)$ but do not intersect between distance 0 and distance π. By assumption, reversing these geodesic segements, i.e., tracing

them backwards from $\alpha(p)$, yields Σ-geodesics (which are no longer necessarily unit speed). Moreover, all of these geodesics remain disjoint until they pass through p, at which point, they all converge.

However, again by Proposition 3, the unit speed geodesics leaving $\alpha(p)$ remain disjoint for distances between 0 and π and they all converge on $\alpha\bigl(\alpha(p)\bigr)$ at distance π.

It follows that $\alpha\bigl(\alpha(p)\bigr)$ must be p. In other words, α^2 is the identity. $\square$

Remark 9. The converse of Proposition 7 does not hold. The $K \equiv 1$ examples provided by Theorem 3 of [6] that are based on Guillemin's Zoll metrics have all their geodesics closed of length 2π (and hence α^2 is the identity), but none of the non-Riemannian ones are geodesically reversible.

3.3. **Geodesic reversibility implies projective flatness.** The next step is to consider the space of *unoriented* Σ-geodesics on M. This only makes sense if one assumes that Σ is geodesically reversible, so assume this for the rest of this subsection.

For each oriented Σ-geodesic $\gamma : S^1 \to M$, let $\beta(\gamma)$ denote the reversed curve, reparametrized so as to be a Σ-geodesic. Obviously $\beta : \Lambda \to \Lambda$ is a fixed-point free involution of Λ, so that the quotient manifold Λ/β is diffeomorphic to $\mathbb{RP}^2$.

Proposition 8. *The path geometry on Λ defined by the geodesics of $[\nabla]$ is invariant under β and hence descends to a well-defined path geometry on Λ/β. Moreover, this path geometry is the path geometry of a projective connection on Λ/β with all of its geodesics closed.*

Proof. Since, by definition, a point p in M lies on a geodesic γ if and only if it lies on $\beta(\gamma)$, it follows that β carries each $[\nabla]$-geodesic into itself. In particular, even though β may not (indeed, most likely does not) preserve ∇, it must preserve $[\nabla]$ since the projective equivalence class of ∇ is determined by its geodesics. Thus, the claims of the Proposition are verified. $\square$

It is at this point that the crucial contribution of LeBrun and Mason [15] enters:

Theorem 2 (LeBrun-Mason)**.** *Any projective structure on $\mathbb{RP}^2$ that has all of its geodesics closed is projectively equivalent to the standard* (*i.e., flat*) *projective structure.*

Corollary 3. *If Σ is a geodesically reversible Finsler structure on $M \simeq S^2$ that satisfies $K \equiv 1$, then the induced projective structure $[\nabla]$ on Λ is projectively flat.* $\square$

Remark 10 (LeBrun and Mason's classification). The article [15] contains, in addition to Theorem 2, much information about *Zoll projective structures* on the 2-sphere, i.e., projective structures on the 2-sphere all of whose geodesics are closed. It turns out that, in a certain sense, there are many more of them than there are Zoll metrics on the 2-sphere.

Their results could quite likely be very useful in understanding the case of non-reversible Finsler metrics satisfying $K \equiv 1$ on the 2-sphere that satisfy $\alpha^2 = \mathrm{id}$, which is still not very well understood. It is even possible that an orbifold version of their results could be useful in the case in which α^2 is not the identity but has finite order. This may be the subject of a later article.

4. Classification

In this final section, the main theorem will be proved.

4.1. **Consequences of projective flatness.** Recall from Section 2 of [5] that if a projective structure on a surface is projectively flat then its dual path geometry is projective and, moreover, projectively flat.

Proposition 9. *If Σ is a geodesically reversible Finsler structure on $M \simeq S^2$ with $K \equiv 1$, then the Σ-geodesics in M are the geodesics of a flat projective structure.*

Proof. The dual path geometry of Λ with its projective structure $[\nabla]$ is M with the space of paths being the Σ-geodesics. Now apply Corollary 3. □

Corollary 4. *Let M be diffeomorphic to S^2. Up to diffeomorphism, any geodesically reversible Finsler structure $\Sigma \subset TM$ with $K \equiv 1$ is equivalent to a member of the* 2*-parameter family described in Theorem* 10 *of* [5].

Proof. In light of Proposition 9, one can apply Theorems 9 and 10 of [5], which gives the result. □

Remark 11. It is interesting to note that each member of the 2-parameter family described in Theorem 10 of [5] is projectively flat and hence geodesically reversible.

4.2. **Reversibility.** Now for the main rigidity theorem.

Theorem 3. *Any reversible Finsler structure on $M \simeq S^2$ that satisfies $K \equiv 1$ is Riemannian and hence isometric to the standard unit sphere.*

Proof. Such a Finsler structure would be geodesically reversible and hence, by Corollary 4, a member of the family described in Theorem 10 of [5]. However, by inspection, the only member of this geodesically reversible family that is actually reversible is the Riemannian one. □

Remark 12 (The argument of Foulon-Reissman). In Section 4 of [11], P. Foulon sketches an argument, due to himself and A. Reissman, that a reversible Finsler metric on the 2-sphere satisfying $K \equiv 1$ that satisfies a certain integral-geometric condition (called by them 'Radon-Gelfand') is necessarily Riemannian. Their condition holds, in particular, whenever the projective structure $[\nabla]$ on Λ is projectively flat. Thus, an alternate proof of Theorem 3 could be given by combining LeBrun and Mason's Theorem 2 with Foulon and Reissman's argument.

The proof of Theorem 3 in this article instead relies on the classification in [5].

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